One-ended subforests and whatnot

Part 5

Let G be a loc. fin. aperiodic ($:=$ every component is infinite) pmp graph on a standard probability space (X,μ).

Recall. The isoperimetric constant of G is

$$\varphi^X_\mu(G) := \inf_{A \subseteq X} \frac{\mu(\partial_G A)}{\mu(A)},$$

where $\partial_G A := \{x \in X \setminus A : x \, G \, A\}$ and A ranges over positively measured Borel finite-component sets for G, i.e. $G|_A$ is component-finite.

(3) **Kaimanovich – Elek.** Not μ-hyperfinite $\iff$ $\exists$ pos.-measured Borel B s.t. $\varphi(G|_B) > 0$.

(4) **Jenna presented.** $\varphi(G) > 0 \xRightarrow{\text{local-global bridge (pmp)}}$ G has exponential growth, in fact $\forall^\mu x \in X \; \forall r$, $|B^G_r(x)| \gtrsim (1+\varphi(G))^r$.

Main theorem (about pmp one-ended subforests). Let G be a loc. fin. aperiodic pmp graph. If G is nowhere 2-ended, then G admits a Borel a.e. 1-ended spanning subforest. And the converse is also true.

Characterizations of μ-hyperfiniteness.

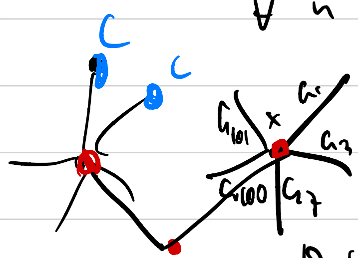
Def. Let G be a Borel graph (X,μ). A finitizing (vertex) cut for G is a set $C \subseteq X$ s.t. $G - C := G|_{X \setminus C}$ is component finite.
The finitizing μ-price for G is

$$fp_\mu(G) := \inf_C \mu(C),$$

where $C \subseteq X$ ranges over all Borel finitizing cuts for G.

99% lemma. Let G be a (null-preserving, i.e. $[\text{null}]_{E_G}$ is null) Borel loc. fin. graph on (X,μ). G is hyperfinite $\iff$ $\mathrm{fp}_\mu(G)=0$. In other words, G restricted to 99% percent of the space is component-finite.

Proof. $\Rightarrow$. Let $G=\bigcup\!\uparrow G_n$ s.t. G_n is component finite. Then (by local finiteness) $\forall x\in X$, $\forall^\infty n$ $(G_n)_x = G_x$. Thus $\exists n$ s.t. for 99% of $x\in X$, we have $(G_n)_x=G_x$. Let's denote by C the complement (of that). Then C is a finitizing cut and C is a 1% set.



Details: Let $X_n := \{x\in X : (G_n)_x = G_x\}$. Then $G_n\nearrow G$
$\Rightarrow X_n \nearrow X$ then $\mu(X_n)\nearrow \mu(X)=1$.

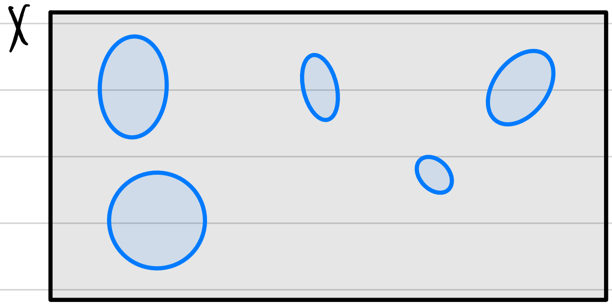
$\Leftarrow$. Just Borel–Cantelli. Suppose $\mathrm{fp}_\mu(G)=0$ so $\exists\,(C_n)$ s.t. $\mu(C_n)\to 0$ and each C_n is a finitizing cut for G. These C_n may not be decreasing, so let's make them so: $C'_m := \bigcup_{n\ge m} C_n$, now each C'_m is still a finitizing cut for G

but also the C'_m are decreasing. We could've chosen the C_n so that $\sum_n \mu(C_n)<\infty$, and then $\mu(C'_m)\le \sum_{n\ge m}\mu(C_n)\to 0$ as $m\to\infty$. Hence $\bigcap_m C'_m$ is null, so may assume it's $\emptyset$ by null-preservingness.
Then $G_n := G - C'_n$ is component-finite and $G=\bigcup\!\uparrow_n G_n$. $\square$

Kaimanovich–Elek theorem. A loc. finite null-preserving Borel graph G on (X,μ) is μ-hyperfinite $\iff$ $\forall$ Borel $B\subseteq X$ of positive measure,
$$\varphi^B_{\mu|_B}(G|_B)=0.$$

Proof. $\Rightarrow$. G is μ-hyperfinite $\Rightarrow$ $G|_B$ is still hyperfinite $\overset{99\%\text{ lemma}}{\Longrightarrow}$ $\exists$ finitizing Borel cut $C\subseteq B$ for $G|_B$. Letting $A:=B\setminus C$, then $G|_A$ is component-finite and $\partial_{G|_B}A\subseteq C$. Hence,
$\mu(\partial_{G|_B}A)/\mu(A)\le 1\%/99\% < \varepsilon$.

$\Leftarrow$. Suppose $\varphi(G|_B) = 0$ $\forall B$. Need to construct an arbitrarily small finitizing cut.



A s.t. $\dfrac{\mu(\partial A)}{\mu(A)} < \varepsilon$.

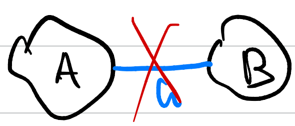
When we take one finite component set A for G, then removing ∂A makes the G-components of $x \in A$ finite. But the components of $x \in X \setminus (A \cup \partial A)$ may still be infinite. So it's enough to find A s.t. $X \setminus (A \cup \partial A) = \emptyset$ and ∂A is small, so ∂A is a small finitizing cut.

Let $\mathcal{A}$ be a maximal collection of Borel subsets of X of positive measure s.t.

(i) each $A \in \mathcal{A}$ is a finite-component set for G

(ii) $\mu(\partial_G \bigcup \mathcal{A}) \leq \varepsilon \cdot \mu(\bigcup \mathcal{A})$.

(iii) any $A, B \in \mathcal{A}$ distinct are disjoint and G-inadjacent

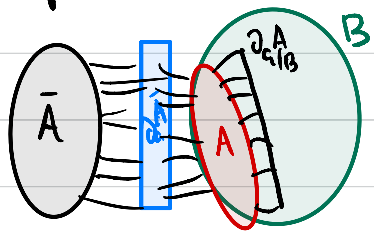


Such $\mathcal{A}$ exists by Zorn's lemma (= measure exhaustion). Because $\mathcal{A}$ is a disjoint collection of pos. measured sets, it must be ctbl since $\mu(X) < \infty$. Thus, $\bar{A} := \bigcup \mathcal{A}$ is Borel. $\partial \bar{A} = \bigcup_{A \in \mathcal{A}} \partial_G A$ by (iii), so $\bar{A}$ is finite-component for G.

$$\text{Also, } \mu(\partial \bar{A}) \leq \sum_{A \in \mathcal{A}} \mu(\partial_G A) \leq \varepsilon \sum_{A \in \mathcal{A}} \mu(A) \overset{(iii)}{=} \varepsilon \cdot \mu(\bar{A}).$$

It remains to show that $\partial_G \bar{A} = X \setminus \bar{A}$ because then $\partial \bar{A}$ is a finitizing cut and $\mu(\partial_G \bar{A}) \leq \varepsilon \cdot \mu(\bar{A}) \leq \varepsilon$. Suppose towards a contradiction that $B := X \setminus (\bar{A} \cup \partial_G \bar{A})$ is not null.

By the hypothesis, $\exists\, A \subseteq B$ of positive measure s.t. $G|_A$ is component-finite and $\mu(\partial_{G|_B} A) \leq \varepsilon \cdot \mu(A)$.



Then $\partial_G A \subseteq \partial_G \bar{A} \cup \partial_{G|_B} A$, so put $\mathcal{A}' := \mathcal{A} \cup \{A\}$.

$$\mu(\partial_G \cup \mathcal{A}') \leq \mu(\partial_G \bar{A}) + \mu(\partial_{G|_B} A) \leq \varepsilon \cdot \mu(\bar{A}) + \varepsilon \cdot \mu(A) = \varepsilon \cdot \mu(\cup \mathcal{A}').$$

Hence $\mathcal{A}'$ contradicts the maximality of $\mathcal{A}$. $\square$